

Jacobi polynomials on the Bernstein ellipse

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Abstract

In this paper, we are concerned with Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ on the Bernstein ellipse with motivation mainly coming from recent studies of convergence rate of spectral interpolation. An explicit representation of $P_n^{(\alpha, \beta)}(x)$ is derived in the variable of parametrization. This formula further allows us to show that the maximum value of $|P_n^{(\alpha, \beta)}(z)|$ over the Bernstein ellipse is attained at one of the endpoints of the major axis if $\alpha + \beta \geq -1$. For the minimum value, we are able to show that for a large class of Gegenbauer polynomials (i.e., $\alpha = \beta$), it is attained at two endpoints of the minor axis. These results particularly extend those previously known only for some special cases. Moreover, we obtain a more refined asymptotic estimate for Jacobi polynomials on the Bernstein ellipse.

Keywords: spectral method, Jacobi polynomials, Bernstein ellipse, extrema, asymptotic estimate

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1 Introduction

Spectral collocation method is a classical and powerful tool to solve integral and differential equations. Suppose that the equation is defined on a finite interval $[-1, 1]$, the basic idea of this approach is to approximate the solution of the equation by its polynomial interpolant of the form

$$f(x) \approx p_n(x) = \sum_{k=1}^n f(x_k) \ell_k(x), \quad -1 \leq x \leq 1, \quad (1.1)$$

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where $\{x_k\}_{k=1}^n$ is a set of distinct nodes and

$$\ell_k(x) = \prod_{\substack{j=1 \\ j \neq k}}^n \frac{x - x_j}{x_k - x_j}, \quad 1 \leq k \leq n,$$

are the Lagrange fundamental polynomials. The values $\{f(x_k)\}_{k=1}^n$ are determined by requiring that the interpolant $p_n(x)$ satisfies the equation exactly at the nodes $\{x_k\}_{k=1}^n$. To ensure rapid convergence of the spectral collocation method, the interpolation nodes $\{x_k\}_{k=1}^n$ with the distribution of density $(1 - x^2)^{-1/2}$ are preferable, and the ideal candidates are the zeros or extrema of classical orthogonal polynomials such as Gegenbauer polynomials, or more generally, the Jacobi polynomials; cf. [8, 14, 18]. The interpolation procedure described above is also known as spectral interpolation.

As is well known, the accuracy of spectral interpolation depends on the regularity of the underlying function $f(x)$, with exponential rate if $f(x)$ is analytic in a neighborhood containing the interval $[-1, 1]$; we refer to [13, 17, 19, 21, 22, 23, 24] for relevant results and [20] for fast implementation. To this end, it is worthwhile to recall that the starting point of these proofs is the so-called Hermite integral formula. More precisely, let \mathcal{E}_ρ be the Bernstein ellipse:

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} \mid z = \frac{1}{2}(u + u^{-1}), \quad u = \rho e^{i\theta}, \quad \rho \geq 1, \quad 0 \leq \theta < 2\pi \right\}. \quad (1.2)$$

The Bernstein ellipse \mathcal{E}_ρ has the foci at ± 1 with the major and minor semi-axes given by $\frac{1}{2}(\rho + \rho^{-1})$ and $\frac{1}{2}(\rho - \rho^{-1})$, respectively. Suppose that $f(x)$ is analytic on and within \mathcal{E}_ρ for some $\rho > 1$, it follows from the Hermite integral formula [2, Theorem 3.6.1] that

$$f(x) - p_n(x) = \frac{1}{2\pi i} \oint_{\mathcal{E}_\rho} \frac{\omega_n(x)f(z)}{\omega_n(z)(z - x)} dz, \quad (1.3)$$

where

$$\omega_n(x) = d_n(x - x_1)(x - x_2) \cdots (x - x_n)$$

with d_n being a positive normalization constant. This in turn implies that

$$|f(x) - p_n(x)| \leq \frac{ML(\mathcal{E}_\rho)}{2\pi d} \max_{\substack{x \in [-1, 1] \\ z \in \mathcal{E}_\rho}} \left| \frac{\omega_n(x)}{\omega_n(z)} \right|, \quad (1.4)$$

where $M = \max_{z \in \mathcal{E}_\rho} |f(z)|$, $L(\mathcal{E}_\rho)$ denotes the length of the circumference of \mathcal{E}_ρ , and d is the distance from \mathcal{E}_ρ to the interval $[-1, 1]$.

For polynomial interpolation at the Jacobi points, i.e., the nodes $\{x_k\}_{k=1}^n$ are the roots of n -th Jacobi polynomial $P_n^{(\alpha, \beta)}(x)$ ($\alpha, \beta > -1$) and the polynomial $\omega_n(z)$ in (1.3) is taken to be $P_n^{(\alpha, \beta)}(z)$, the properties of $P_n^{(\alpha, \beta)}(x)$ on the Bernstein ellipse are essential in the analysis of convergence rate. For the Gegenbauer polynomials $C_n^\lambda(z)$

(corresponding to the special cases $\alpha = \beta = \lambda - \frac{1}{2}$ of the Jacobi polynomials), by noting that (see [22, Lemma 3.1])

$$C_n^\lambda(z) = \sum_{k=0}^n g_k^\lambda g_{n-k}^\lambda u^{n-2k}, \quad z = \frac{1}{2}(u + u^{-1}), \quad n \geq 0, \quad (1.5)$$

where

$$g_0^\lambda = 1, \quad g_k^\lambda = \binom{k+\lambda-1}{k} = \frac{\Gamma(k+\lambda)}{k!\Gamma(\lambda)}, \quad 1 \leq k \leq n,$$

the following asymptotic estimate of Gegenbauer polynomials on the Bernstein ellipse is obtained by Xie, Wang and Zhao in [22, Theorem 3.2]: there exists $0 < \varepsilon \leq \frac{1}{2}$ such that

$$\left| (1 - u^{-2})^{-\lambda} - \frac{C_n^\lambda(z)}{g_n^\lambda u^n} \right| \leq A(\rho, \lambda) n^{\varepsilon-1} + \mathcal{O}(n^{-1}), \quad z \in \mathcal{E}_\rho, \quad (1.6)$$

for $\rho > 1$, $\lambda > -\frac{1}{2}$ and $\lambda \neq 0$, where

$$A(\rho, \lambda) = |1 - \lambda| \left| (1 - \rho^{-2})^{-\lambda} - 1 \right|.$$

The estimate (1.6) plays an important role in the rigorous proofs of exponential convergence of Gegenbauer interpolation and spectral differentiation conducted in [22]. Later, in a paper regarding superconvergence of Jacobi-Gauss type spectral interpolation [21], Wang, Zhao and Zhang have made use of the following estimate of the lower bound for Jacobi polynomial on the Bernstein ellipse:

$$\min_{z \in \mathcal{E}_\rho} \left| P_n^{(\alpha, \beta)}(z) \right| \geq C(\rho; \alpha, \beta) n^{-\frac{1}{2}} \rho^{n+1} (1 + \mathcal{O}(n^{-1})), \quad (1.7)$$

where $C(\rho; \alpha, \beta) = \min_{|u|=\rho} |\phi_0(u; \alpha, \beta)|$ is a constant independent of n , and the function $\phi_0(u; \alpha, \beta)$ is regular for $|u| = \rho > 1$, and $|u| = 1$ but $u \neq \pm 1$; see [21, Equation (4.7)]. This result follows directly from the asymptotic formula of Jacobi polynomials [16, Theorem 8.21.9]. We note that, however, except for the very special cases like $\alpha = \beta = -\frac{1}{2}$, the explicit form of $C(\rho; \alpha, \beta)$ is not available.

We also note that there is a close connection between polynomial interpolation and the potential theory [18, Chapter 5]. More specifically, let us define the discrete potential function associated with the nodes $\{x_k\}_{k=1}^n$ by

$$E_n(z) = \frac{1}{n} \sum_{k=1}^n \log |z - x_k|.$$

This function is harmonic in the complex plane except at $\{x_k\}_{k=1}^n$ and can be viewed as the potential generated by all $\{x_k\}_{k=1}^n$ if each x_k is interpreted as a point charge of strength $1/n$ and the repulsion is inverse-linear. Clearly, we have $|\omega_n(z)| = e^{nE_n(z)}$. Thus, if we choose x_k to be the zeros of $P_n^{(\alpha, \beta)}(x)$, then the extrema of $|P_n^{(\alpha, \beta)}(z)|$ implies the extrema of the corresponding potential $E_n(z)$ as well.

It is the aim of the present research to conduct more complete studies of Jacobi polynomials on the Bernstein ellipse, including the explicit formula, extrema of the absolute value and the asymptotic estimate. Our main contributions are listed below:

- An explicit formula of $P_n^{(\alpha, \beta)}(x)$ is derived in the variable of parametrization, which generalizes (1.5) valid for the Gegenbauer case.
- The extrema of $|P_n^{(\alpha, \beta)}(z)|$ on the Bernstein ellipse \mathcal{E}_ρ are identified under some assumptions on the parameters. We show that the maximum value is attained at one of the endpoints of the major axis if $\alpha + \beta \geq -1$. This particularly extends [9, Theorem 4.5.1] established by Ismail, which is valid for the Gegenbauer polynomials. For the minimum value, we are able to show that for a large class of Gegenbauer polynomials, it is attained at two endpoints of the minor axis.
- We provide a more refined and computable asymptotic estimate as well as a lower bound for the Jacobi polynomials on the Bernstein ellipse, which generalizes (1.6) concerning the Gegenbauer polynomials.

The rest of this paper is organized as follows. We first give a brief review of Jacobi polynomials in Section 2, which includes some basic properties that will be used later. Section 3 is devoted to the explicit representation of Jacobi polynomials on the Bernstein ellipse. The extrema of Jacobi polynomials on the Bernstein ellipse are discussed in Section 4. The identification of maximum value relies on a three-term recurrence relation for the coefficients arising in the explicit formula. For the minimum value, we first deal with the Chebyshev polynomials of the first and second kinds and then extend the results to Gegenbauer polynomials. The asymptotic estimate and the lower bound of Jacobi polynomials on the Bernstein ellipse are presented in Section 5.

2 Some properties of Jacobi polynomials

In this section, we collect some basic properties of Jacobi polynomials which will be used in the subsequent analysis. All these properties can be found in the classical book of Szegő [16].

Let $P_n^{(\alpha, \beta)}(x)$ denote the Jacobi polynomial of degree n , which is defined explicitly by

$$P_n^{(\alpha, \beta)}(x) = 2^{-n} \sum_{k=0}^n \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (x-1)^k (x+1)^{n-k}, \quad \alpha, \beta > -1. \quad (2.1)$$

The Jacobi polynomials are orthogonal over $[-1, 1]$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$, that is,

$$\int_{-1}^1 (1-x)^\alpha (1+x)^\beta P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) dx = h_n^{(\alpha, \beta)} \delta_{m,n}, \quad (2.2)$$

where $\delta_{m,n}$ is the Kronecker delta and

$$h_n^{(\alpha, \beta)} = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}.$$

From the explicit formula (2.1), it is easily seen that

$$P_n^{(\alpha, \beta)}(x) = k_n^{(\alpha, \beta)} x^n + \dots, \quad (2.3)$$

where the leading coefficient $k_n^{(\alpha, \beta)}$ is given by

$$k_n^{(\alpha, \beta)} = \frac{1}{2^n} \binom{2n + \alpha + \beta}{n} = \frac{\Gamma(2n + \alpha + \beta + 1)}{2^n n! \Gamma(n + \alpha + \beta + 1)}. \quad (2.4)$$

Let $q := \max\{\alpha, \beta\}$ with $\alpha, \beta > -1$. The maximum of $|P_n^{(\alpha, \beta)}(x)|$ on the interval $[-1, 1]$ is given by (see [16, Theorem 7.32.1])

$$\max_{x \in [-1, 1]} |P_n^{(\alpha, \beta)}(x)| = \begin{cases} \binom{n+q}{n}, & \text{if } q \geq -\frac{1}{2}, \\ |P_n^{(\alpha, \beta)}(\tilde{x})|, & \text{if } q < -\frac{1}{2}, \end{cases} \quad (2.5)$$

where \tilde{x} is one of the two maximum points nearest $(\beta - \alpha)/(\alpha + \beta + 1)$. Indeed, when $q \geq -\frac{1}{2}$, the maximum of $|P_n^{(\alpha, \beta)}(x)|$ is attained at one of the endpoints $\{-1, 1\}$.

When $\alpha = \beta$, the Jacobi polynomials are, up to some positive constants, also known as Gegenbauer (or the ultraspherical) polynomials $C_n^\lambda(x)$. More precisely, we have

$$C_n^\lambda(x) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(2\lambda)} \frac{\Gamma(n + 2\lambda)}{\Gamma(n + \lambda + \frac{1}{2})} P_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(x), \quad \lambda > -\frac{1}{2}. \quad (2.6)$$

The orthogonality of Gegenbauer polynomials reads

$$\int_{-1}^1 C_m^\lambda(x) C_n^\lambda(x) (1 - x^2)^{\lambda - 1/2} dx = h_n^\lambda \delta_{m, n},$$

where $h_n^\lambda = \frac{2^{1-2\lambda} \pi \Gamma(n+2\lambda)}{\Gamma^2(\lambda) n! (n+\lambda)}$. Since the weight function $(1 - x^2)^{\lambda - 1/2}$ is an even function, it is readily seen that following symmetry relations hold:

$$C_n^\lambda(x) = (-1)^n C_n^\lambda(-x), \quad n \geq 0. \quad (2.7)$$

Thus, $C_n^\lambda(x)$ is an even function for even n and an odd function for odd n .

The Chebyshev polynomials of the first and second kinds are

$$T_n(\cos \theta) = \cos(n\theta), \quad U_n(\cos \theta) = \frac{\sin((n+1)\theta)}{\sin \theta}, \quad n \geq 0,$$

respectively. When $z \in \mathcal{E}_\rho$, the Chebyshev polynomials have the following simple representations in the variable of parametrization:

$$T_n(z) = \frac{1}{2}(u^n + u^{-n}), \quad U_n(z) = \frac{u^{n+1} - u^{-n-1}}{u - u^{-1}}. \quad (2.8)$$

They are special cases of Gegenbauer polynomials, and the relations are given by

$$T_n(x) = \lim_{\lambda \rightarrow 0} \frac{n}{2} \frac{C_n^\lambda(x)}{\lambda}, \quad n \geq 1; \quad U_n(x) = C_n^1(x), \quad n \geq 0. \quad (2.9)$$

Equivalently, one has

$$T_n(x) = \frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})} P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x), \quad U_n(x) = \frac{\Gamma(n+2)\Gamma(\frac{3}{2})}{\Gamma(n+\frac{3}{2})} P_n^{(\frac{1}{2}, \frac{1}{2})}(x). \quad (2.10)$$

Finally, let $1 > x_1^\lambda > x_2^\lambda > \dots > x_n^\lambda > -1$ be the zeros of Gegenbauer polynomials. By [16, Theorem 6.21.1], it follows that

$$\frac{\partial x_j^\lambda}{\partial \lambda} < 0, \quad j = 1, \dots, \lfloor n/2 \rfloor, \quad (2.11)$$

where $\lfloor x \rfloor$ denotes the integer part of x . Thus, the positive zeros of a Gegenbauer polynomial $C_n^\lambda(x)$ strictly decrease with respect to the parameter λ .

3 An explicit formula of Jacobi polynomials on the Bernstein ellipse

It is the aim of this section to prove the following theorem, which gives an explicit representation of $P_n^{(\alpha, \beta)}(x)$ on the Bernstein ellipse in the variable of parametrization.

Theorem 3.1. *For $z \in \mathcal{E}_\rho$, i.e.,*

$$z = \frac{1}{2} (u + u^{-1}), \quad |u| = \rho \geq 1, \quad (3.1)$$

we have

$$P_n^{(\alpha, \beta)}(z) = \sum_{k=-n}^n d_{|k|, n} u^k, \quad (3.2)$$

where the coefficients are given by

$$d_{k, n} = \frac{(n + \alpha + \beta + 1)_k (k + \alpha + 1)_{n-k}}{(n - k)! 2^{2k} \Gamma(k + 1)} \times {}_3F_2 \left[\begin{matrix} k - n, & n + k + \alpha + \beta + 1, & k + \frac{1}{2}; \\ & k + \alpha + 1, & 2k + 1; \end{matrix} \right], \quad (3.3)$$

for $0 \leq k \leq n$.

To show Theorem 3.1, we note that Chebyshev polynomials of the first kind $T_n(x)$ has a simple explicit formula (2.8) on the Bernstein ellipse, the strategy is then to expand Jacobi polynomials in terms of $T_n(x)$. The connection formula between two different families of Jacobi polynomials are stated in the following lemma (see [1, Theorem 7.1.1]).

Lemma 3.2. *Assume that*

$$P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^n c_{n,k} P_k^{(\gamma, \delta)}(x). \quad (3.4)$$

Then the connection coefficients are given by

$$c_{n,k} = \frac{(n + \alpha + \beta + 1)_k (k + \alpha + 1)_{n-k} (2k + \gamma + \delta + 1) \Gamma(k + \gamma + \delta + 1)}{(n - k)! \Gamma(2k + \gamma + \delta + 2)} \\ \times {}_3F_2 \left[\begin{matrix} k - n, n + k + \alpha + \beta + 1, k + \gamma + 1; \\ k + \alpha + 1, 2k + \gamma + \delta + 2; \end{matrix} 1 \right], \quad (3.5)$$

where

$$(a)_0 = 1, \quad (a)_k = \frac{\Gamma(a + k)}{\Gamma(a)} = a(a + 1) \cdots (a + k - 1), \quad k \geq 1, \quad (3.6)$$

is the Pochhammer symbol and

$${}_3F_2 \left[\begin{matrix} a_1, a_2, a_3; \\ b_1, b_2; \end{matrix} z \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k (a_3)_k}{(b_1)_k (b_2)_k} \frac{z^k}{k!} \quad (3.7)$$

is the generalized hypergeometric function.

With the aid of Lemma 3.2, we are now ready to prove Theorem 3.1.

Proof of Theorem 3.1 By taking $\gamma = \delta = -\frac{1}{2}$ in Lemma 3.2, it follows that

$$P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^n {}_1' \frac{(n + \alpha + \beta + 1)_k (k + \alpha + 1)_{n-k} 2\Gamma(k + 1)}{(n - k)! \Gamma(2k + 1)} \\ \times {}_3F_2 \left[\begin{matrix} k - n, n + k + \alpha + \beta + 1, k + \frac{1}{2}; \\ k + \alpha + 1, 2k + 1; \end{matrix} 1 \right] P_k^{(-\frac{1}{2}, -\frac{1}{2})}(x), \quad (3.8)$$

where the prime indicates that the first term of the sum should be halved. This, together with the identity (2.10), gives

$$P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^n {}_1' \frac{(n + \alpha + \beta + 1)_k (k + \alpha + 1)_{n-k} 2\Gamma(k + 1)}{(n - k)! \Gamma(2k + 1)} \\ \times {}_3F_2 \left[\begin{matrix} k - n, n + k + \alpha + \beta + 1, k + \frac{1}{2}; \\ k + \alpha + 1, 2k + 1; \end{matrix} 1 \right] \frac{\Gamma(k + \frac{1}{2})}{\Gamma(k + 1) \Gamma(\frac{1}{2})} T_k(x) \\ = \sum_{k=0}^n {}_1' \frac{(n + \alpha + \beta + 1)_k (k + \alpha + 1)_{n-k}}{(n - k)! 2^{2k-1} \Gamma(k + 1)} \\ \times {}_3F_2 \left[\begin{matrix} k - n, n + k + \alpha + \beta + 1, k + \frac{1}{2}; \\ k + \alpha + 1, 2k + 1; \end{matrix} 1 \right] T_k(x), \quad (3.9)$$

where we have made use of the duplication formula (cf. [12, Formula 5.5.5])

$$\Gamma(2z) = \pi^{-1/2} 2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right), \quad 2z \neq 0, -1, -2, \dots, \quad (3.10)$$

in the second step.

Note that $T_k(z)$ has a simple expression $\frac{1}{2}(u + u^{-n})$ on \mathcal{E}_ρ . Substituting this formula into the last equation of (3.9) gives us the desired result.

This completes the proof of Theorem 3.1. \square

Remark 3.3. Suppose that $u = e^{i\theta}$ (i.e., $\rho = 1$), we obtain from (3.2) the following trigonometric representations of Jacobi polynomials

$$P_n^{(\alpha, \beta)}(\cos \theta) = d_{0,n} + 2 \sum_{k=1}^n d_{k,n} \cos(k\theta). \quad (3.11)$$

The above formula seems to be new, except for the special case $\alpha = \beta$ (cf. [16, Formula (4.9.19)]).

Remark 3.4. When $\alpha = \beta$, the coefficients $d_{k,n}$ can be further simplified with the help of the properties of hypergeometric function ${}_3F_2$. Indeed, on account of the fact (see [1, Theorem 3.5.5]) that

$${}_3F_2 \left[\begin{matrix} a, b, c; \\ (a+b+1)/2, 2c; \end{matrix} 1 \right] = \frac{\Gamma(\frac{1}{2}) \Gamma(c + \frac{1}{2}) \Gamma(\frac{a+b+1}{2}) \Gamma(c - \frac{a+b-1}{2})}{\Gamma(\frac{a+1}{2}) \Gamma(\frac{b+1}{2}) \Gamma(c - \frac{a-1}{2}) \Gamma(c - \frac{b-1}{2})}, \quad (3.12)$$

it follows from (3.3) and straightforward calculations that

$$d_{k,n} = \begin{cases} \frac{2^{2\alpha} \Gamma(n + \alpha + 1) \Gamma(\frac{k+n+1}{2} + \alpha) \Gamma(\frac{n-k+1}{2} + \alpha)}{\sqrt{\pi} \Gamma(n + 2\alpha + 1) \Gamma(\frac{k+n}{2} + 1) \Gamma(\frac{n-k}{2} + 1) \Gamma(\alpha + \frac{1}{2})}, & \text{if } n - k \text{ is even,} \\ 0, & \text{if } n - k \text{ is odd.} \end{cases} \quad (3.13)$$

This particularly implies that

$$P_n^{(\alpha, \alpha)}(z) = \sum_{k=0}^n d_{|n-2k|,n} u^{n-2k}.$$

Up to some constant factors, this recovers (1.5) which was derived via the three-term recurrence relation of Gegenbauer polynomials in [22]. The approach used therein, however, seems difficult to be generalized to handle the Jacobi case.

4 Extrema of Jacobi polynomials on the Bernstein ellipse

In this section, we will consider the extrema of Jacobi polynomials on the Bernstein ellipse. The maximum value and the minimum value will be discussed in subsections 4.1 and 4.2, respectively.

4.1 Maximum value

By [9, Theorem 4.5.1], it is known that for the Gegenbauer polynomials $C_n^\lambda(x)$, $\max |C_n^\lambda(z)|$, $z \in \mathcal{E}_\rho$ with $\lambda \geq 0$ is attained at the right endpoint of the major axis. It comes out that similar property holds for the Jacobi polynomials $P_n^{(\alpha, \beta)}$ with $\alpha + \beta \geq -1$, which is our main result of this section.

Theorem 4.1. *For $\rho \geq 1$ and $n \geq 1$, we have*

- (i) *If $\alpha > \beta$ and $\alpha + \beta \geq -1$, then the maximum value of $|P_n^{(\alpha, \beta)}(z)|$ is attained uniquely at the right endpoint of the major axis, i.e.,*

$$\max_{z \in \mathcal{E}_\rho} |P_n^{(\alpha, \beta)}(z)| = P_n^{(\alpha, \beta)}\left(\frac{1}{2}(\rho + \rho^{-1})\right). \quad (4.1)$$

- (ii) *If $\alpha < \beta$ and $\alpha + \beta \geq -1$, then the maximum value of $|P_n^{(\alpha, \beta)}(z)|$ is attained uniquely at the left endpoint of the major axis, i.e.,*

$$\max_{z \in \mathcal{E}_\rho} |P_n^{(\alpha, \beta)}(z)| = |P_n^{(\alpha, \beta)}\left(-\frac{1}{2}(\rho + \rho^{-1})\right)|. \quad (4.2)$$

- (iii) *If $\alpha = \beta \geq -1/2$, then the maximum value of $|P_n^{(\alpha, \beta)}(z)|$ is attained at two endpoints of the major axis, i.e.,*

$$\max_{z \in \mathcal{E}_\rho} |P_n^{(\alpha, \beta)}(z)| = |P_n^{(\alpha, \beta)}\left(\pm \frac{1}{2}(\rho + \rho^{-1})\right)|. \quad (4.3)$$

Moreover, the maximum value can only be attained at these two real points $\pm \frac{1}{2}(\rho + \rho^{-1})$ if $\alpha = \beta > -1/2$.

The assertion in item (iii) corresponds to the case of Gegenbauer polynomials mentioned at the very beginning. Moreover, the condition $\alpha + \beta \geq -1$ implies that $\max\{\alpha, \beta\} \geq -\frac{1}{2}$. By setting $\rho = 1$ in the above theorem, we recover the result concerning maximum of $|P_n^{(\alpha, \beta)}(x)|$ over the orthogonal interval $[-1, 1]$, as explained after (2.5).

The proof of Theorem 4.1 relies on the explicit formula of $P_n^{(\alpha, \beta)}(z)$ on the Bernstein ellipse established in Theorem 3.1. The essential issue here is to determine the signs of the coefficients $d_{k,n}$ appearing in (3.2) under various conditions on the parameters α and β ; see Proposition 4.4 below. To proceed, we start with the following proposition which reveals a recurrence relation for the coefficients $\{d_{k,n}\}_{k=0}^n$ and plays a fundamental role in the sequel.

Proposition 4.2. *With $d_{k,n}$ defined in (3.3), we have, for each $k \geq 0$ and $k + 2 \leq n$,*

$$\begin{aligned} d_{k,n} = & \frac{2(\alpha - \beta)(k + 1)}{n(n + \alpha + \beta + 1) - k^2 - (\alpha + \beta + 1)k} d_{k+1,n} \\ & + \frac{n(n + \alpha + \beta + 1) - (k + 2)^2 + (\alpha + \beta + 1)(k + 2)}{n(n + \alpha + \beta + 1) - k^2 - (\alpha + \beta + 1)k} d_{k+2,n}, \end{aligned} \quad (4.4)$$

with initial conditions

$$d_{n,n} = \frac{\Gamma(2n + \alpha + \beta + 1)}{2^{2n}\Gamma(n + \alpha + \beta + 1)\Gamma(n + 1)}, \quad d_{n-1,n} = \frac{(\alpha - \beta)\Gamma(2n + \alpha + \beta)}{2^{2n-1}\Gamma(n + \alpha + \beta + 1)\Gamma(n)}. \quad (4.5)$$

Proof. In view of (3.9), it is readily seen that

$$P_n^{(\alpha,\beta)}(x) = d_{0,n} + 2 \sum_{k=1}^n d_{k,n} T_k(x). \quad (4.6)$$

We recall from [16, Theorem 4.2.1] that the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ satisfies the following linear differential equation

$$(1 - x^2)y''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]y'(x) + n(n + \alpha + \beta + 1)y(x) = 0.$$

Substituting (4.6) into the above equation gives

$$\begin{aligned} & 2 \sum_{k=1}^n d_{k,n} \{ (1 - x^2)T_k''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]T_k'(x) \\ & \quad + n(n + \alpha + \beta + 1)T_k(x) \} + n(n + \alpha + \beta + 1)d_{0,n} = 0. \end{aligned} \quad (4.7)$$

Our strategy now is to rewrite the left hand side of (4.7) in terms of the Chebyshev polynomial of the second kind $U_k(x)$. To this end, note that $T_k(x)$ satisfies

$$(1 - x^2)T_k''(x) - xT_k'(x) + k^2T_k(x) = 0,$$

we then obtain from (4.7) that

$$\begin{aligned} & 2 \sum_{k=1}^n d_{k,n} \{ [\beta - \alpha - (\alpha + \beta + 1)x]T_k'(x) \\ & \quad + [n(n + \alpha + \beta + 1) - k^2]T_k(x) \} + n(n + \alpha + \beta + 1)d_{0,n} = 0. \end{aligned}$$

This, together with the facts (cf. [12, §18.9]) that

$$\begin{aligned} T_k'(x) &= kU_{k-1}(x), \\ 2xU_k(x) &= U_{k+1}(x) + U_{k-1}(x), \quad k \geq 1, \\ 2T_k(x) &= U_k(x) - U_{k-2}(x), \quad k \geq 1 \text{ with } U_{-1}(x) = 0, \end{aligned}$$

implies

$$\begin{aligned} & \sum_{k=1}^n d_{k,n} \{ [n(n + \alpha + \beta + 1) - k^2 - k(\alpha + \beta + 1)]U_k(x) + 2(\beta - \alpha)kU_{k-1}(x) \\ & \quad - [n(n + \alpha + \beta + 1) - k^2 + k(\alpha + \beta + 1)]U_{k-2}(x) \} + n(n + \alpha + \beta + 1)d_{0,n} = 0. \end{aligned}$$

By setting the coefficients of $U_k(x)$, $1 \leq k \leq n - 2$ and the constant term to be zero, the recurrence relation (4.4) is immediate.

This completes the proof of Proposition 4.2. \square

Remark 4.3. From (4.4), it is readily seen that if $\alpha = \beta$, the three-term recurrence relation can be simplified as

$$d_{k,n} = \frac{n(n+2\alpha+1) - (k+2)^2 + (2\alpha+1)(k+2)}{n(n+2\alpha+1) - k^2 - (2\alpha+1)k} d_{k+2,n}. \quad (4.8)$$

In addition, note that the coefficients $d_{k,n}$ in (3.3) involve the hypergeometric function ${}_3F_2$, it would be helpful to use the recurrence relation (4.4) in actual computations.

As a consequence of Proposition 4.2, we are able to determine the signs of the coefficients $\{d_{k,n}\}_{k=0}^n$ in the following proposition.

Proposition 4.4. *For $0 \leq k \leq n$, we have*

- (i) *If $\alpha > \beta$ and $\alpha + \beta \geq -1$, then $d_{k,n} > 0$.*
- (ii) *If $\alpha < \beta$ and $\alpha + \beta \geq -1$, then $(-1)^{n-k} d_{k,n} > 0$.*
- (iii) *If $\alpha = \beta > -\frac{1}{2}$, then*

$$d_{k,n} \begin{cases} > 0, & \text{if } n-k \text{ is even,} \\ = 0, & \text{if } n-k \text{ is odd.} \end{cases} \quad (4.9)$$

If $\alpha = \beta = -\frac{1}{2}$, then

$$d_{k,n} \begin{cases} > 0, & \text{if } k = n, \\ = 0, & \text{if } k = 1, 2, \dots, n-1. \end{cases} \quad (4.10)$$

Proof. If $\alpha > \beta$ and $\alpha + \beta \geq -1$, it is easily seen that

$$\frac{2(\alpha - \beta)(k+1)}{n(n + \alpha + \beta + 1) - k^2 - (\alpha + \beta + 1)k} = \frac{2(\alpha - \beta)(k+1)}{n^2 - k^2 + (\alpha + \beta + 1)(n - k)} > 0,$$

and

$$\begin{aligned} & \frac{n(n + \alpha + \beta + 1) - (k+2)^2 + (\alpha + \beta + 1)(k+2)}{n(n + \alpha + \beta + 1) - k^2 - (\alpha + \beta + 1)k} \\ &= \frac{n^2 - (k+2)^2 + (\alpha + \beta + 1)(n + k + 2)}{n^2 - k^2 + (\alpha + \beta + 1)(n - k)} > 0, \end{aligned} \quad (4.11)$$

for $0 \leq k \leq n-2$. These, together with the recurrence relation (4.4) and the fact that both of the initial values $d_{n-1,n}$ and $d_{n,n}$ are positive (see (4.5)), imply the assertion in item (i).

Similarly, if $\alpha < \beta$ and $\alpha + \beta \geq -1$, we have

$$\frac{2(\alpha - \beta)(k+1)}{n(n + \alpha + \beta + 1) - k^2 - (\alpha + \beta + 1)k} < 0,$$

and

$$\frac{n(n + \alpha + \beta + 1) - (k + 2)^2 + (\alpha + \beta + 1)(k + 2)}{n(n + \alpha + \beta + 1) - k^2 - (\alpha + \beta + 1)k} > 0,$$

for $0 \leq k \leq n - 2$. Since $d_{n-1,n} < 0$ and $d_{n,n} > 0$ in this case, we again obtain from (4.4) that $\{d_{k,n}\}_{k=0}^n$ is an alternating sequence, as required.

Finally, if $\alpha = \beta \geq -\frac{1}{2}$, the assertion in item (iii) follows immediately from (3.13).

This completes the proof of Proposition 4.4. \square

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1 If $\alpha > \beta$ and $\alpha + \beta \geq -1$, we recall from item (i) in Proposition 4.4 that $d_{k,n} > 0$ for $0 \leq k \leq n$. On account of (3.1) and (3.2), it is straightforward to see that

$$\left| P_n^{(\alpha, \beta)}(z) \right| = \left| \sum_{k=-n}^n d_{|k|,n} u^k \right| \leq \sum_{k=-n}^n d_{|k|,n} \rho^k = P_n^{(\alpha, \beta)} \left(\frac{1}{2} (\rho + \rho^{-1}) \right).$$

Thus, $\max_{z \in \mathcal{E}_\rho} \left| P_n^{(\alpha, \beta)}(z) \right|$ can be achieved if and only if $u = \rho$, which is (4.1).

Similarly, if $\alpha < \beta$ and $\alpha + \beta \geq -1$, a combination of item (ii) in Proposition 4.4 and (3.2) implies that

$$\begin{aligned} \left| P_n^{(\alpha, \beta)}(z) \right| &= \left| \sum_{k=-n}^n d_{|k|,n} u^k \right| \leq \sum_{k=-n}^n (-1)^{n-|k|} d_{|k|,n} \rho^k \\ &= (-1)^n P_n^{(\alpha, \beta)} \left(-\frac{1}{2} (\rho + \rho^{-1}) \right) = \left| P_n^{(\alpha, \beta)} \left(-\frac{1}{2} (\rho + \rho^{-1}) \right) \right|. \end{aligned}$$

Hence, the maximum value can be achieved if and only if $u = -\rho$, as shown in (4.2).

To show (4.3), we see from Remark 3.4, (4.9) and (4.10) that

$$\left| P_n^{(\alpha, \beta)}(z) \right| = \left| \sum_{k=0}^n d_{|n-2k|,n} u^{n-2k} \right| \leq \sum_{k=0}^n d_{|n-2k|,n} \rho^{n-2k}.$$

Thus, the maximum value can be achieved when $u = \pm \rho$ and (4.3) follows. Moreover, if $\alpha = \beta > -\frac{1}{2}$, since $d_{|n-2k|,n}$ is strictly positive (see (4.9)) for $0 \leq k \leq n$, the maximum value can only be achieved at two endpoints of the major axis.

This completes the proof of Theorem 4.1. \square

Remark 4.5. For the very special case $\alpha = \beta = -\frac{1}{2}$, the Jacobi polynomials (up to a normalization constant) are the Chebyshev polynomials of the first kind $T_n(x)$; see (2.10). In this case, we obtain from (2.8) that for $z \in \mathcal{E}_\rho$,

$$\begin{aligned} |T_n(z)| &= \frac{1}{2} \sqrt{\rho^{2n} + \rho^{-2n} + 2 \cos(2n\theta)} \\ &\leq \frac{1}{2} \sqrt{\rho^{2n} + \rho^{-2n} + 2} = \frac{1}{2} (\rho^n + \rho^{-n}). \end{aligned} \tag{4.12}$$

It is then clear that $\max_{z \in \mathcal{E}_\rho} |T_n(z)|$ is attained if and only if $\cos(2n\theta) = 1$, i.e., at $2n$ points

$$\hat{z}_{n,k} = \frac{1}{2} \left(\rho e^{i \frac{k\pi}{n}} + \left(\rho e^{i \frac{k\pi}{n}} \right)^{-1} \right), \quad k = 0, \dots, 2n-1. \quad (4.13)$$

4.2 Minimum value

The identification of minimum value of $|P_n^{(\alpha,\beta)}(z)|$ on the Bernstein ellipse \mathcal{E}_ρ is, in general, much more involved than that of its maximum value. Since the minimum value will be zero when $\rho = 1$, we will restrict our attention to $\rho > 1$ and focus on the ultraspherical case $\alpha = \beta$ in this section. In view of the relations (2.6) and (2.10), the results will be presented in terms of $T_n(x)$, $U_n(x)$ and $C_n^\lambda(x)$.

To provide some intuition about the location where Gegenbauer polynomials attain the minimum value, we perform some numerical experiments of $|C_n^\lambda(z)|$ with $z \in \mathcal{E}_\rho$; see Figures 1–3 for different choices of the parameters λ , n and ρ . Note that we omit the numerical results for $-\frac{1}{2} < \lambda < 0$ and even $n \geq 2$, since they are similar to those shown in Figure 3. The numerical studies imply that the minimum value depends on the parameters λ , n , ρ , and further suggest the following conjecture concerning the observations.

Conjecture 4.1. *It is conjectured that*

- (i) *If $\lambda > 0$ and $n \geq 1$ is odd, $\min_{z \in \mathcal{E}_\rho} |C_n^\lambda(z)|$ is attained at $\pm \frac{i}{2}(\rho - \rho^{-1})$ for $\rho > 1$, i.e., at two endpoints of the minor axis.*
- (ii) *If $\lambda > 0$ and $n \geq 2$ is even, there exists a critical value $\varrho(n, \lambda)$ depending on the parameters n and λ such that $\min_{z \in \mathcal{E}_\rho} |C_n^\lambda(z)|$ is attained at $\pm \frac{i}{2}(\rho - \rho^{-1})$ for $\rho \geq \varrho(n, \lambda)$.*
- (iii) *If $-\frac{1}{2} < \lambda < 0$ and $n \geq 2$, there exists a critical value $\tilde{\varrho}(n, \lambda)$ depending on the parameters n and λ such that $\min_{z \in \mathcal{E}_\rho} |C_n^\lambda(z)|$ is attained at $\pm \frac{i}{2}(\rho + \rho^{-1})$ for $\rho \geq \tilde{\varrho}(n, \lambda)$, i.e., at two endpoints of the major axis.*

In what follows, we shall prove items (i) and (ii) of the above conjecture under the assumptions that $\rho \geq \frac{1}{2}(\sqrt{2} + \sqrt{6}) \approx 1.932$ and $\lambda \geq 1$ (for item (ii)).

We first deal with the Chebyshev polynomials of the first and second kinds. The relevant results, on one hand, will provide important insights for the general case, on the other hand they are crucial in further analysis.

Theorem 4.6 (Minimum value of Chebyshev polynomials of the first kind $T_n(z)$). *For $\rho > 1$ and $n \geq 1$, we have*

$$\min_{z \in \mathcal{E}_\rho} |T_n(z)| = \frac{1}{2} (\rho^n - \rho^{-n}), \quad (4.14)$$

and the minimum value is attained at $2n$ points

$$\tilde{z}_{n,k} = \frac{1}{2} \left(\rho e^{i \frac{2k+1}{2n}\pi} + \left(\rho e^{i \frac{2k+1}{2n}\pi} \right)^{-1} \right), \quad k = 0, \dots, 2n-1. \quad (4.15)$$

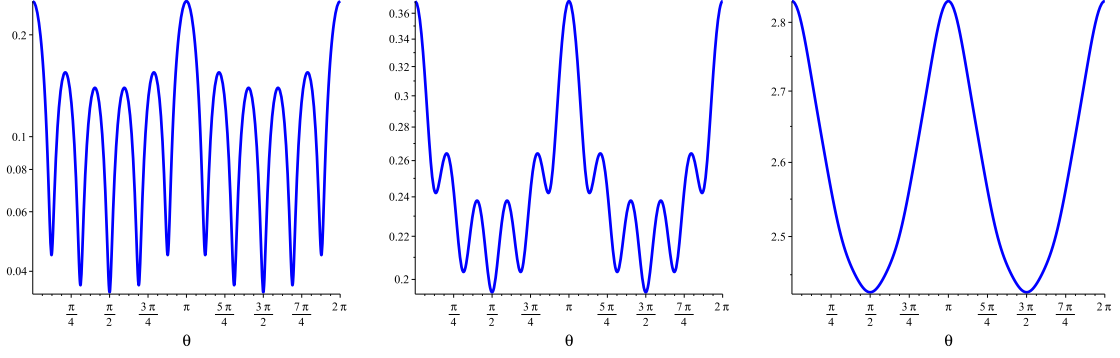


Figure 1: Plot of $|C_5^{1/4}(z)|$ with $z = \frac{1}{2}(\rho e^{i\theta} + \rho^{-1}e^{-i\theta}) \in \mathcal{E}_\rho$ for $\rho = 1.05$ (left), $\rho = 1.25$ (middle) and $\rho = 2$ (right). Here θ ranges from 0 to 2π .

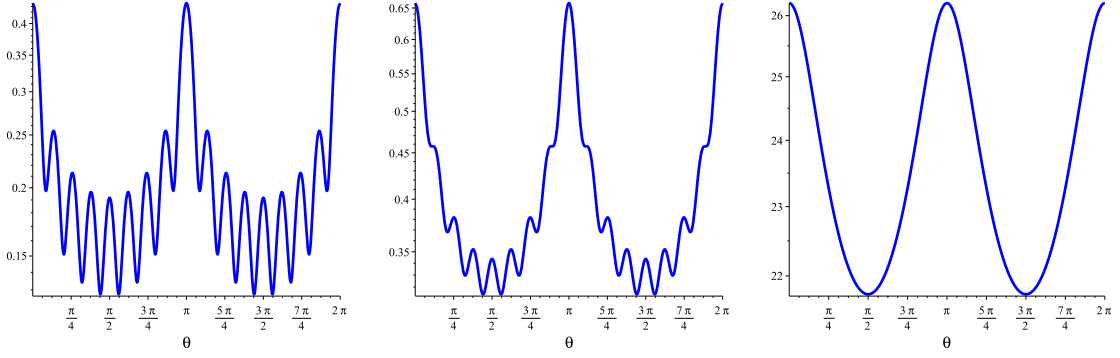


Figure 2: Plot of $|C_8^{1/3}(z)|$ with $z = \frac{1}{2}(\rho e^{i\theta} + \rho^{-1}e^{-i\theta}) \in \mathcal{E}_\rho$ for $\rho = 1.1$ (left), $\rho = 1.2$ (middle) and $\rho = 2$ (right). Here θ ranges from 0 to 2π .

Proof. From (4.12) it is readily seen that $|T_n(z)|$ attains its minimum value if and only if

$$\cos(2n\theta) = -1,$$

i.e., for $\theta = \frac{2k+1}{2n}\pi$ with $k = 0, \dots, 2n-1$. The desired results follow immediately. \square

Remark 4.7. From (4.15), it follows that, for odd n , the minimum value of $T_n(z)$ can be attained at $\pm \frac{i}{2}(\rho - \rho^{-1})$.

We next proceed to the Chebyshev polynomial of the second kind $U_n(z)$. The following lower bound can be found in [11, formula (1.53)]:

$$|U_n(z)| \geq \frac{\rho^{n+1} - \rho^{-n-1}}{\rho + \rho^{-1}}, \quad z \in \mathcal{E}_\rho \text{ with } \rho > 1. \quad (4.16)$$

Our next theorem shows that this lower bound is attainable only when n is odd. If n is even, a new and attainable lower bound will be presented under some conditions on ρ .

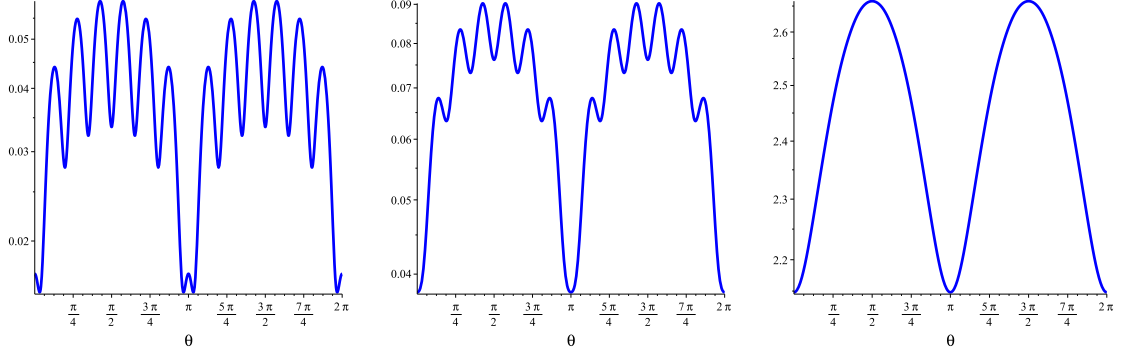


Figure 3: Plot of $|C_7^{-1/3}(z)|$ with $z = \frac{1}{2}(\rho e^{i\theta} + \rho^{-1}e^{-i\theta}) \in \mathcal{E}_\rho$ for $\rho = 1.1$ (left), $\rho = 1.2$ (middle) and $\rho = 2$ (right). Here θ ranges from 0 to 2π .

Theorem 4.8 (Minimum value of Chebyshev polynomials of the second kind $U_n(z)$).
For $n \geq 1$, we have

$$\min_{z \in \mathcal{E}_\rho} |U_n(z)| = \begin{cases} \frac{\rho^{n+1} - \rho^{-n-1}}{\rho + \rho^{-1}}, & \text{if } n \text{ is odd and } \rho > 1, \\ \frac{\rho^{n+1} + \rho^{-n-1}}{\rho + \rho^{-1}}, & \text{if } n \text{ is even and } \rho \geq \rho_n^*, \end{cases} \quad (4.17)$$

where $\rho_n^* > 1$ is the unique root of the equation

$$a_{n+1}(\rho) - (n+1)a_1(\rho) = 0, \quad (4.18)$$

and where

$$a_k(\rho) = \frac{1}{2} (\rho^k + \rho^{-k}), \quad k \geq 0. \quad (4.19)$$

Moreover, in both cases the minimum value is attained if and only if $z = \pm \frac{i}{2} (\rho - \rho^{-1})$, i.e., at two endpoints of the minor axis.

The above theorem can actually be seen from a remarkable connection between $U_n(z)$ and the kernel $K_n(z)$ arising in the contour integral representation of the remainder term of an n -point Gauss quadrature for the Chebyshev weight function of the second kind. More precisely, let f be an analytic function on and within the Bernstein ellipse \mathcal{E}_ρ . The Gaussian quadrature rule for the Chebyshev weight function of the second kind reads

$$\int_{-1}^1 f(t)(1-t^2)^{1/2} dt = \sum_{k=1}^n \lambda_k^{(n)} f(\tau_k^{(n)}) + R_n(f), \quad (4.20)$$

where $\tau_k^{(n)} = \cos(k\pi/(n+1))$ are the zeros of the Chebyshev polynomial of the second kind $U_n(z)$, and $\lambda_k^{(n)}$ are the corresponding Christoffel numbers. The remainder term

$R_n(f)$ admits the following contour representation:

$$R_n(f) = \frac{1}{2\pi i} \oint_{\mathcal{E}_\rho} K_n(z) f(z) dz, \quad (4.21)$$

where the kernel $K_n(z)$ is given by

$$K_n(z) = \frac{q_n(z)}{U_n(z)} = \frac{\pi}{u^{n+1}U_n(z)}, \quad z = \frac{1}{2}(u + u^{-1}), \quad |u| = \rho,$$

where $q_n(z) = \int_{-1}^1 \frac{U_n(t)(1-t^2)^{1/2}}{z-t} dt$ and the second equality follows from [7, Equation 3.613.3]. The above formula particularly implies that $|U_n(z)|^2$ is proportional to the reciprocal of $|K_n(z)|^2$ if $z \in \mathcal{E}_\rho$. Since the maximum value of $|K_n(z)|$ over \mathcal{E}_ρ has been studied in the context of estimating the remainder term R_n in [6, 5], Theorem 4.8 follows directly from [6, Theorem 5.2] and [5, Theorem 1]. For completeness, we include a more direct proof in what follows.

Proof of Theorem 4.8 By (2.8), it is readily seen that

$$|U_n(z)|^2 = \frac{a_{2n+2}(\rho) - \cos((2n+2)\theta)}{a_2(\rho) - \cos(2\theta)}, \quad z = \frac{1}{2}(\rho e^{i\theta} + \rho^{-1}e^{-i\theta}) \in \mathcal{E}_\rho. \quad (4.22)$$

Denote by $\varphi_n(\theta)$ the function appearing on the right hand side of (4.22). It is then equivalent to consider the minimum of $\varphi_n(\theta)$ for $\theta \in [0, 2\pi]$.

We start with the easy case that n is odd. By (4.22), it is clear that

$$\varphi_n(\theta) \geq \frac{a_{2n+2}(\rho) - 1}{a_2(\rho) + 1} = \left(\frac{\rho^{n+1} - \rho^{-n-1}}{\rho + \rho^{-1}} \right)^2,$$

and the lower bound on the right hand side is attained if and only if $\cos((2n+2)\theta) = 1$ and $\cos(2\theta) = -1$, which gives $\theta = \frac{\pi}{2}$ or $\frac{3}{2}\pi$. Thus, the minimum can only be attained at two endpoints of the minor axis, i.e., at the points $z = \pm \frac{i}{2}(\rho - \rho^{-1})$, as desired.

If n is even, a straightforward calculation shows that

$$\begin{aligned} \varphi_n(\theta) - \varphi_n\left(\frac{\pi}{2}\right) &= \frac{a_{2n+2}(\rho) - \cos((2n+2)\theta)}{a_2(\rho) - \cos(2\theta)} - \frac{a_{2n+2}(\rho) + 1}{a_2(\rho) + 1} \\ &= \frac{2(\cos \theta)^2}{a_2(\rho) - \cos(2\theta)} \left[\frac{a_{2n+2}(\rho) + 1}{a_2(\rho) + 1} - \left(\frac{\cos((n+1)\theta)}{\cos \theta} \right)^2 \right] \\ &= \frac{2(\cos \theta)^2}{a_2(\rho) - \cos(2\theta)} \left[\left(\frac{a_{n+1}(\rho)}{a_1(\rho)} \right)^2 - \left(\frac{\cos((n+1)\theta)}{\cos \theta} \right)^2 \right]. \end{aligned} \quad (4.23)$$

To this end, we note that, on one hand,

$$\left| \frac{\cos(n+1)\theta}{\cos \theta} \right| = \left| \frac{\sin(n+1)(\frac{\pi}{2} - \theta)}{\sin(\frac{\pi}{2} - \theta)} \right| = |U_n(t)|,$$

where $t = \cos\left(\frac{\pi}{2} - \theta\right)$. It then follows from the property of $U_n(z)$ that

$$\max_{\theta \in [0, 2\pi]} \left| \frac{\cos(n+1)\theta}{\cos \theta} \right| = n+1, \quad (4.24)$$

and the upper bound can be attained if and only if $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. On the other hand, it is easily seen that the function $a_{n+1}(\rho)/a_1(\rho)$ is strictly increasing for $\rho \in [1, \infty)$ and n fixed. Hence, if $\rho \geq \rho_n^*$, we see from (4.23) and (4.24) that

$$\varphi_n(\theta) - \varphi_n\left(\frac{\pi}{2}\right) \geq 0.$$

In addition, since

$$\varphi_n\left(\frac{\pi}{2}\right) = \varphi_n\left(\frac{3\pi}{2}\right) = \frac{a_{2n+2}(\rho) + 1}{a_2(\rho) + 1} = \left(\frac{\rho^{n+1} + \rho^{-n-1}}{\rho + \rho^{-1}}\right)^2,$$

the second case in (4.17) follows. It is also easy to see that the minimum is attained if and only if $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

This completes the proof of Theorem 4.8. \square

Remark 4.9. For even n and $1 < \rho < \rho_n^*$, one can conclude from [5, Theorem 1] that the minimum value of $|U_n(z)|$ is attained at some $z^* = \frac{1}{2}(\rho e^{i\theta^*} + \rho^{-1}e^{-i\theta^*})$ with $\theta^* \in (\frac{n}{n+1}\frac{\pi}{2}, \frac{\pi}{2})$, which is slightly off the imaginary axis. Moreover, from [5, Theorem 2] we know that $\{\rho_n^*\}_{n=1}^\infty$ is a strictly decreasing sequence and $\rho_n^* \rightarrow 1$ as $n \rightarrow \infty$; see Figure 4 for an illustration.

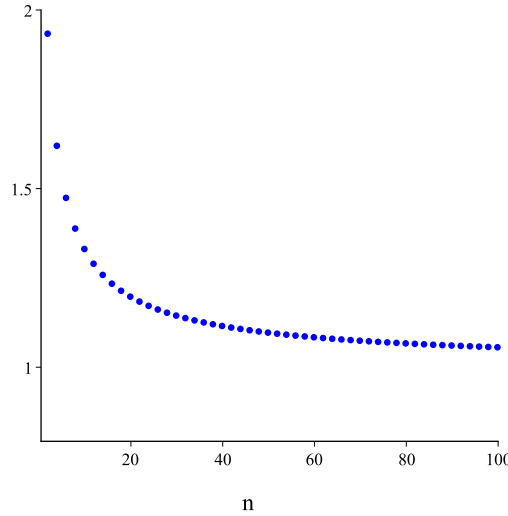


Figure 4: Plot of the sequence $\{\rho_n^*\}$ for $n = 2, 4, \dots, 100$.

We finally come to the Gegenbauer polynomials $C_n^\lambda(x)$. Besides the trivial case[§] $n = 1$, we have the following theorem.

[§]If $n = 1$, $C_1^\lambda(x) = 2\lambda x$. Thus, the minimum value of $|C_1^\lambda(z)|$ can only be attained at two endpoints of the minor axis $\pm \frac{i}{2}(\rho - \rho^{-1})$.

Theorem 4.10 (Minimum value of Gegenbauer polynomials $C_n^\lambda(z)$). *Let $\rho_2^* = \frac{1}{2}(\sqrt{2} + \sqrt{6}) \approx 1.932$ be the unique root of (4.18) with $n = 2$. For $\rho \geq \rho_2^*$ and $n \geq 2$, the minimum value of $|C_n^\lambda(z)|$ is attained at two endpoints of the minor axis, i.e.,*

$$\min_{z \in \mathcal{E}_\rho} |C_n^\lambda(z)| = |C_n^\lambda(\pm \frac{i}{2}(\rho - \rho^{-1}))|, \quad (4.25)$$

provided $\lambda > 1$, or $0 < \lambda < 1$ and n is odd.

We precede the proof of Theorem 4.10 with the following lemma.

Lemma 4.11. *Let $z \in \mathcal{E}_\rho$ and define*

$$\mathcal{R}(z) = \frac{z^2 - s^2}{z^2 - t^2},$$

where $s, t \in (0, 1)$. Then, for $s > t$ and $\rho \geq \rho_2^*$,

$$\max_{z \in \mathcal{E}_\rho} |\mathcal{R}(z)| = |\mathcal{R}(\pm \frac{i}{2}(\rho - \rho^{-1}))|. \quad (4.26)$$

Proof. See [15, Lemma 4.1]. □

Proof of Theorem 4.10 Let $\{x_j^\lambda\}_{j=1}^n$ be the zeros of $C_n^\lambda(x)$ arranged in decreasing order. The symmetry relation (2.7) implies that

$$C_n^\lambda(z) = k_n^\lambda \prod_{k=1}^n (z - x_k^\lambda) = k_n^\lambda z^{n-2\lfloor n/2 \rfloor} \prod_{k=1}^{\lfloor n/2 \rfloor} (z^2 - (x_k^\lambda)^2), \quad (4.27)$$

where k_n^λ is the leading coefficient of $C_n^\lambda(x)$. Moreover, we see that $x_k^\lambda > 0$ for $k = 1, \dots, \lfloor n/2 \rfloor$.

Let $0 < y_{\lfloor n/2 \rfloor} < \dots < y_1 < 1$ be the positive zeros of the Chebyshev polynomials of the second kind $U_n(z)$. Again, we could rewrite $U_n(z)$ as

$$U_n(z) = 2^n z^{n-2\lfloor n/2 \rfloor} \prod_{k=1}^{\lfloor n/2 \rfloor} (z^2 - y_k^2). \quad (4.28)$$

By (2.11), it follows that

$$0 < x_k^\lambda < y_k < 1, \quad (4.29)$$

for $k = 1, \dots, \lfloor n/2 \rfloor$.

To find $\min_{z \in \mathcal{E}_\rho} |C_n^\lambda(z)|$ is equivalent to find $\max_{z \in \mathcal{E}_\rho} \left| \frac{1}{C_n^\lambda(z)} \right|$. A combination of (4.27) and (4.28) gives

$$\left| \frac{1}{C_n^\lambda(z)} \right| = \frac{2^n}{k_n^\lambda} \prod_{k=1}^{\lfloor n/2 \rfloor} \left| \frac{z^2 - y_k^2}{z^2 - (x_k^\lambda)^2} \right| \times \left| \frac{1}{U_n(z)} \right|. \quad (4.30)$$

In view of (4.29), Lemma 4.11, Theorem 4.8 and the monotonicity of ρ_n^* aforementioned in Remark 4.9, we conclude that all the terms on the right hand side of (4.30) attain their maximum values at $z = \pm \frac{i}{2}(\rho - \rho^{-1})$ for $\rho \geq \rho_2^*$. Therefore, $|C_n^\lambda(z)|$ attains its minimum at two endpoints of the minor axis provided $\lambda > 1$.

The case for $0 < \lambda < 1$ and odd n can be proved in a similar manner. We only need to replace $U_n(z)$ in (4.30) by the Chebyshev polynomials of the first kind $T_n(z)$, and to make use of Remark 4.7 instead. The details are left to the interested readers.

This completes the proof of Theorem 4.10. \square

5 Asymptotic estimate of Jacobi polynomials on the Bernstein ellipse

From the explicit formula (1.5) of Gegenbauer polynomials on the Bernstein ellipse, the authors in [22] derive an asymptotic estimate of $C_n^\lambda(z)$ as shown in (1.6). Due to the complexity of the coefficients $d_{k,n}$ given in (3.3), it is difficult to apply the same approach to obtain the asymptotic estimate of Jacobi polynomials on the Bernstein ellipse.

To this end, we note that a more computable form has been given in [10], where the authors actually consider asymptotics of polynomials orthogonal with respect to a modified Jacobi weight function

$$w(x) = (1-x)^\alpha(1+x)^\beta h(x), \quad (5.1)$$

with $\alpha, \beta > -1$ and $h(x)$ being real analytic and strictly positive on $[-1, 1]$. Based on the Riemann-Hilbert (RH) approach [4], various asymptotics of the monic/orthonormal polynomials in the complex plane have been derived in [10], which in particular includes a full asymptotic expansion for the monic polynomials outside $[-1, 1]$.

To state the relevant results, we need the function

$$\varphi(z) = z + \sqrt{z^2 - 1}, \quad z \in \mathbb{C} \setminus [-1, 1], \quad (5.2)$$

where $\sqrt{z^2 - 1}$ is analytic in $\mathbb{C} \setminus [-1, 1]$ and behaves like z as $z \rightarrow \infty$. This function is a conforming mapping from $\mathbb{C} \setminus [-1, 1]$ onto the exterior of the unit circle. Thus,

$$|\varphi(z)| > 1, \quad z \in \mathbb{C} \setminus [-1, 1].$$

As in [10], we also define the Szegő function of a weight w by

$$D(z) = D(z; w) = \exp \left(\frac{(z^2 - 1)^{1/2}}{2\pi} \int_{-1}^1 \frac{\log w(x)}{\sqrt{1-x^2}} \frac{dx}{z-x} \right), \quad z \in \mathbb{C} \setminus [-1, 1], \quad (5.3)$$

and

$$D_\infty = \lim_{z \rightarrow \infty} D(z) = \exp \left(\frac{1}{2\pi} \int_{-1}^1 \frac{w(x)}{\sqrt{1-x^2}} dx \right). \quad (5.4)$$

The function $D(z)$ is analytic for $z \in \mathbb{C} \setminus [-1, 1]$.

Let $\pi_n(x)$ denote the monic orthogonal polynomial of degree n associated with (5.1). It is shown in [10, Theorem 1.4] that $\pi_n(z)$ has an asymptotic expansion of the form

$$\pi_n(z) \sim \frac{D_\infty}{D(z)} \frac{\varphi(z)^{n+\frac{1}{2}}}{2^{n+\frac{1}{2}}(z^2-1)^{\frac{1}{4}}} \left[1 + \sum_{k=1}^{\infty} \frac{\Pi_k(z)}{n^k} \right], \quad n \rightarrow \infty, \quad (5.5)$$

uniformly valid for z in any compact subsets of $\mathbb{C} \setminus [-1, 1]$. The functions $\Pi_k(z)$, which are analytic on $z \in \mathbb{C} \setminus [-1, 1]$, are rational in φ . They are explicitly computable via the RH approach but with more complicated form as k increases. The first two terms are

$$\Pi_1(z) = -\frac{4\alpha^2 - 1}{8(\varphi(z) - 1)} + \frac{4\beta^2 - 1}{8(\varphi(z) + 1)}, \quad (5.6)$$

and

$$\begin{aligned} \Pi_2(z) = & \frac{(4\alpha^2 - 1)(\alpha + \beta)}{16(\varphi(z) - 1)} - \frac{(4\beta^2 - 1)(\alpha + \beta)}{16(\varphi(z) + 1)} - \frac{(4\alpha^2 - 1)(4\beta^2 - 1)}{128(z^2 - 1)} \\ & + \frac{2\alpha^2 + 2\beta^2 - 5}{64} \left[\frac{4\alpha^2 - 1}{(\varphi(z) - 1)^2} + \frac{4\beta^2 - 1}{8(\varphi(z) + 1)^2} \right]. \end{aligned} \quad (5.7)$$

For an efficient numerical calculations of the higher-order terms $\Pi_k(z)$, we refer to recent work [3].

Obviously, the classical Jacobi polynomials correspond to the case $h(x) = 1$ in (5.1). We then have the following asymptotic estimate of Jacobi polynomials on the Bernstein ellipse in the variable of parametrization.

Proposition 5.1. *For $z \in \mathcal{E}_\rho$, i.e.,*

$$z = \frac{1}{2} (u + u^{-1}), \quad u = \rho e^{i\theta}, \quad \rho > 1, \quad 0 \leq \theta \leq 2\pi, \quad (5.8)$$

we have, for large n ,

$$\left| (1 - u^{-1})^{-\alpha-\frac{1}{2}} (1 + u^{-1})^{-\beta-\frac{1}{2}} - \frac{\sqrt{\pi n}}{2^{\alpha+\beta} u^n} P_n^{(\alpha, \beta)}(z) \right| \leq \Lambda(\rho, \alpha, \beta) n^{-1} + \mathcal{O}(n^{-2}). \quad (5.9)$$

where

$$\Lambda(\rho, \alpha, \beta) = \max_{|u|=\rho} \left| \frac{4\hat{\Pi}_1(u) - (\alpha + \beta)^2 - (\alpha + \beta) - \frac{1}{2}}{4(1 - u^{-1})^{\alpha+\frac{1}{2}}(1 + u^{-1})^{\beta+\frac{1}{2}}} \right|. \quad (5.10)$$

and

$$\hat{\Pi}_1(u) = \frac{4\beta^2 - 1}{8(u + 1)} - \frac{4\alpha^2 - 1}{8(u - 1)}. \quad (5.11)$$

Furthermore, the error is uniformly bounded for $z \in \mathcal{E}_\rho$ with $\rho > 1$.

Proof. We first derive the uniform asymptotics of $P_n^{(\alpha, \beta)}(z)$ for $z \in \mathbb{C} \setminus [-1, 1]$. In view of the facts that

$$D(z; (1-x)^\alpha(1+x)^\beta) = \frac{(z-1)^{\alpha/2}(z+1)^{\beta/2}}{\varphi(z)^{(\alpha+\beta)/2}}$$

and

$$D_\infty = \lim_{z \rightarrow \infty} D(z; (1-x)^\alpha(1+x)^\beta) = 2^{-(\alpha+\beta)/2},$$

it then follows from (2.3) and (5.5) that

$$\frac{P_n^{(\alpha, \beta)}(z)}{k_n^{(\alpha, \beta)}} \sim \frac{\varphi(z)^{n+\frac{\alpha+\beta+1}{2}}}{2^{n+\frac{\alpha+\beta+1}{2}}(z-1)^{\frac{\alpha}{2}+\frac{1}{4}}(z+1)^{\frac{\beta}{2}+\frac{1}{4}}} \left[1 + \sum_{k=1}^{\infty} \frac{\Pi_k(z)}{n^k} \right], \quad n \rightarrow \infty, \quad (5.12)$$

uniformly valid for z in any compact subsets of $\mathbb{C} \setminus [-1, 1]$, where $k_n^{(\alpha, \beta)}$ is defined as in (2.4). Using asymptotic formulas for the Gamma functions (see [12, Formulas 5.11.3 and 5.11.13]), we deduce that

$$k_n^{(\alpha, \beta)} = \frac{2^{n+\alpha+\beta}}{\sqrt{\pi n}} \left[1 - \frac{(\alpha+\beta)^2 + (\alpha+\beta) + \frac{1}{2}}{4n} + \mathcal{O}\left(\frac{1}{n^2}\right) \right].$$

This, together with (5.12), implies that

$$\begin{aligned} P_n^{(\alpha, \beta)}(z) &= \frac{2^{\frac{\alpha+\beta}{2}} \varphi(z)^{n+\frac{\alpha+\beta+1}{2}}}{\sqrt{2\pi n} (z-1)^{\frac{\alpha}{2}+\frac{1}{4}} (z+1)^{\frac{\beta}{2}+\frac{1}{4}}} \\ &\quad \times \left[1 + \frac{4\Pi_1(z) - (\alpha+\beta)^2 - (\alpha+\beta) - \frac{1}{2}}{4n} + \mathcal{O}\left(\frac{1}{n^2}\right) \right], \quad n \rightarrow \infty, \end{aligned} \quad (5.13)$$

where $\Pi_1(z)$ is given in (5.11).

If $z \in \mathcal{E}_\rho \subset \mathbb{C} \setminus [-1, 1]$, which can be parameterized through the argument u as shown in (3.1), it is straightforward to check that

$$\begin{aligned} \varphi(z) &= z + \sqrt{z^2 - 1} = u, \\ (z-1)^{\frac{\alpha}{2}+\frac{1}{4}}(z+1)^{\frac{\beta}{2}+\frac{1}{4}} &= \frac{(u-1)^{\alpha+\frac{1}{2}}(u+1)^{\beta+\frac{1}{2}}}{(2u)^{\frac{\alpha+\beta+1}{2}}}. \end{aligned}$$

A combination of the above two formulas and (5.13) gives

$$\begin{aligned} P_n^{(\alpha, \beta)}(z) &= \frac{2^{\alpha+\beta} u^n}{\sqrt{\pi n}} (1-u^{-1})^{-\alpha-\frac{1}{2}} (1+u^{-1})^{-\beta-\frac{1}{2}} \\ &\quad \times \left[1 + \frac{4\hat{\Pi}_1(u) - (\alpha+\beta)^2 - (\alpha+\beta) - \frac{1}{2}}{4n} + \mathcal{O}\left(\frac{1}{n^2}\right) \right], \end{aligned} \quad (5.14)$$

where $\hat{\Pi}_1(u)$ is defined as in (5.11) and the asymptotics is valid uniformly for $z \in \mathcal{E}_\rho$ with $\rho > 1$. By using the above uniform asymptotic, it is straightforward to derive the desired result (5.9) and this completes the proof of Proposition 5.1. \square

Remark 5.2. One should compare the asymptotic estimate (5.9) with (1.6). It is worthwhile to point out that the error in (5.9) is of order $\mathcal{O}(1/n)$. Indeed, a full asymptotic expansion of $P_n^{(\alpha, \beta)}(z)$ in terms of powers of $1/n$ on the Bernstein ellipse \mathcal{E}_ρ can be derived by combining (5.12) and a full asymptotic expansion of the leading coefficient $k_n^{(\alpha, \beta)}$. We note that this form of asymptotic expansion has been mentioned in [16, Theorem 8.21.9], but without explicit formulas for the coefficients.

Remark 5.3. As a direct consequence of Proposition 5.1, we have

$$\lim_{n \rightarrow \infty} \frac{P_n^{(\alpha, \beta)}(z) \sqrt{n\pi}}{2^{\alpha+\beta} u^n} = (1 - u^{-1})^{-\alpha - \frac{1}{2}} (1 + u^{-1})^{-\beta - \frac{1}{2}}, \quad (5.15)$$

where $z = \frac{1}{2}(u + u^{-1})$ and $|u| = \rho > 1$.

A further application of Proposition 5.1 is that we are able to derive the following lower bound for the Jacobi polynomial on the Bernstein ellipse, which particularly implies a more explicit expression of the constant $C(\rho; \alpha, \beta)$ appearing in (1.7).

Corollary 5.4. *For $z = \frac{1}{2}(u + u^{-1}) \in \mathcal{E}_\rho$, we have*

$$\min_{z \in \mathcal{E}_\rho} |P_n^{(\alpha, \beta)}(z)| \geq \frac{C_n(\alpha, \beta) 2^{\alpha+\beta} \pi^{-\frac{1}{2}} \rho^n}{\max_{|u|=\rho} \left| (1 - u^{-1})^{\alpha + \frac{1}{2}} (1 + u^{-1})^{\beta + \frac{1}{2}} \right| \sqrt{n}}, \quad (5.16)$$

where $C_n(\alpha, \beta)$ is a positive constant and $C_n(\alpha, \beta) \sim 1$ for large n . Moreover,

$$\max_{|u|=\rho} \left| (1 - u^{-1})^{\alpha + \frac{1}{2}} (1 + u^{-1})^{\beta + \frac{1}{2}} \right| = \begin{cases} (1 + \rho^{-2})^{\alpha + \frac{1}{2}}, & \text{if } \alpha = \beta \geq -\frac{1}{2}, \\ (1 - \rho^{-2})^{\alpha + \frac{1}{2}}, & \text{if } -1 < \alpha = \beta < -\frac{1}{2}. \end{cases} \quad (5.17)$$

Proof. The lower bound follows immediately from Proposition 5.1 and the elementary inequality $||z_1| - |z_2|| \leq |z_1 - z_2|$. To show (5.17), by setting $u = \rho e^{i\theta}$, $\rho > 1$ and $0 \leq \theta < 2\pi$, it is easily seen that

$$1 - \rho^{-2} \leq |1 - u^{-2}| = \sqrt{1 - 2\rho^{-2} \cos(2\theta) + \rho^{-4}} \leq 1 + \rho^{-2}. \quad (5.18)$$

Hence, if $|u| = \rho$, $|1 - u^{-2}|$ attains its maximum value at $\pm \rho i$ and its minimum value at $\pm \rho$, which gives us (5.17). \square

6 Concluding remarks

In this paper, we have investigated several basic properties of Jacobi polynomials on the Bernstein ellipse, which include the explicit formula, extrema of the absolute values as well as a refined asymptotic estimate. These results provide some further insight into Jacobi polynomials and can be adaptable to some practical applications such as establishing an explicit error bound of the spectral interpolation at the Jacobi nodes.

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